Algebraic Number Theory
Exercise Sheet 11

Prof. Dr. Nikita Geldhauser
PD Dr. Maksim Zhykhovich

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Exercise 1. For every square-free positive integer d compute the group of units $\mathcal{O}_K^*$, where $K = \mathbb{Q}[\sqrt{-d}]$.
Hint: Recall that $x \in K$ is a unit if and only if $x \in \mathcal{O}_K$ and $N_{\mathbb{Q}}^K(x) = \pm 1$ (see Exercise 3, Sheet 2).

Exercise 2. Let ζ_5 be a primitive 5th root of unity, let $K = \mathbb{Q}(\zeta_5)$.
(1) Show that $1 + \zeta_5 \in \mathcal{O}_K^*$.
(2) Let H be the subgroup in $\mathcal{O}_K^*$ generated by $1 + \zeta_5$. Show that $\mathcal{O}_K^*/H$ is finite.

Exercise 3. Let $K = \mathbb{Q}[\sqrt{d}] \subset \mathbb{R}$ be a real quadratic field, where d is a square-free positive integer. Let A be the ring of integers of K.
(0) Show that the subgroup $H = A^* \cap]0, +\infty[$ of A^* is isomorphic to $\mathbb{Z}$. Deduce that there is only one generator of H which is greater than 1. We call this generator the fundamental unit.
(1) Let $v = a + b\sqrt{d} \in A^*$, $a, b \in \mathbb{Q}$. Show that $v > 1$ if and only if $a > 0$ and $b > 0$.
(2) Let $u = \min\{v \in A^* | \, v > 1\}$. Show that u is the fundamental unit of K.
(3) For every positive integer n, let $u^n = a_n + b_n\sqrt{d}$, $a_n, b_n \in \mathbb{Q}$. Show that the sequence $\{b_n\}$ is increasing.
(4) Find the fundamental unit $u > 1$ for $d = 2, 5, 7$.
Hint: Use (3) and the description of A.
(5) Suppose that the fundamental unit $u > 1$ is known. Describe the positive integer solutions of Pell-Fermat equations: $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$
Hint: Consider the two cases: $d \not\equiv 1 \mod 4$ and $d \equiv 1 \mod 4$.
(6) Describe the positive integer solutions of Pell-Fermat equations for $d = 2, 5, 7$.